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COMMENT

Comment on 'Time-like flows of energy momentum and particle trajectories for the Klein–Gordon equation'

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Abstract

Horton *et al* (2000 *J. Phys. A: Math. Gen.* **33** 7337) have proposed Bohm-type particle trajectories accompanying a Klein–Gordon wavefunction ψ on Minkowski space. From two vector fields on space–time, W^+ and W^-, defined in terms of ψ, they intend to construct a time-like vector field W, the integral curves of which are the possible trajectories, by the following rule: at every space–time point, take either $W = W^+$ or $W = W^-$ depending on which is time-like. This procedure, however, is ill-defined as soon as both are time-like, or both space-like. Indeed, they cannot both be time-like, but they can well both be space-like, contrary to the central claim of Horton *et al*. We point out the gap in their proof, we provide a counter example, and we argue that, even for a rather arbitrary wavefunction, the points where both W^+ and W^- are space-like can form a set of positive measure.

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Let $\psi = \mathrm{e}^{P+\mathrm{i}S}$ (where P and S are real) solve the Klein–Gordon equation, $-\Box\psi = m^2\psi$. Set $P_\mu = \partial_\mu P$, $S_\mu = \partial_\mu S$, and

$$\theta = \sinh^{-1}\frac{P^\mu P_\mu - S^\mu S_\mu}{2P^\mu S_\mu}.$$

The case that P_μ and S_μ are orthogonal is an exceptional one that we neglect, just as Horton *et al* [1]. For W_μ, one is supposed to take either $W_\mu^+ = \mathrm{e}^\theta P_\mu + S_\mu$ or $W_\mu^- = -\mathrm{e}^{-\theta} P_\mu + S_\mu$, depending on which is time-like; they cannot both be time-like since they are orthogonal. The question is, could they both be space-like?

Horton *et al* [1] declare that W_μ^+ and W_μ^- cannot both be space-like and argue that otherwise there exists a Lorentz frame such that $W_0^+ = 0$ and $W_0^- = 0$, thus $\mathrm{e}^\theta P_0 = -\mathrm{e}^{-\theta} P_0$. From this, they conclude $\mathrm{e}^\theta = -\mathrm{e}^{-\theta}$, which is impossible.

It is correct that any two orthogonal space-like vectors span a space-like two-plane (corresponding to $x^0 = 0$ in the appropriate Lorentz frame), but no contradiction arises since in this case P_0 would be zero (in this frame). This is the mistake in the proof.

Together with $W_0^+ = 0$ (or $W_0^- = 0$), $P_0 = 0$ implies $S_0 = 0$. Hence, for W_μ^+ and W_μ^- to be space-like it is necessary and sufficient that P_μ and S_μ span a space-like two-plane.

Can this case occur? Clearly, since the Klein–Gordon equation is of second order, one may choose ψ and $\partial_0\psi$ *ad libitum* on the $x^0 = 0$ hyperplane. Can it also occur for the first-order Klein–Gordon equation $-\mathrm{i}\partial_0\psi = \sqrt{m^2 - \Delta}\,\psi$, or, equivalently, for functions from the positive-energy subspace? Here is an example. Let ψ be a superposition of three[1] plane waves

$$\psi(x) = \sum_{i=1}^{3} c_i\, \mathrm{e}^{\mathrm{i}k_\mu^{(i)} x^\mu}$$

with wave vectors $k_\mu^{(1)} = (m, 0, 0, 0)$, $k_\mu^{(2)} = (\sqrt{27}m, \sqrt{26}m, 0, 0)$, $k_\mu^{(3)} = (\sqrt{27}m, 0, \sqrt{26}m, 0)$, and $c_1 = 3$, $c_2 = -1/\sqrt{3} - \mathrm{i}$, $c_3 = \mathrm{i}$. Then, at the coordinate origin, we find $P_\mu = (0, \alpha, -\alpha, 0)$ and $S_\mu = (0, -\beta, 0, 0)$ with $\alpha = \sqrt{26}m/\gamma$, $\beta = \alpha/\sqrt{3}$ and $\gamma = 3 - 1\big/\sqrt{3}$. This example could also be made square-integrable by replacing the plane waves $\exp\left(\mathrm{i}k_\mu^{(i)} x^\mu\right)$ by positive-energy L^2 Klein–Gordon functions $\varphi^{(i)}(x)$ with the properties $\varphi^{(i)}(0) = 1$ and $\partial_\mu\varphi^{(i)}(0) = \mathrm{i}k_\mu^{(i)}$.

One may suspect, however, that perhaps this particular wavefunction ψ is very exceptional, and perhaps even that, for this special wavefunction, the coordinate origin is a rather atypical point, so that the sort of situation just described can be ignored. After all, we would be willing to ignore the case where P_μ and S_μ are orthogonal because in the eight-dimensional space of all possible pairs of vectors P_μ, S_μ it corresponds to a subset of dimension seven, and therefore one would expect that the space–time points where this occurs form a set of measure zero.

But since $W_\mu^+ W^{+\mu}$ and $W_\mu^- W^{-\mu}$ are continuous functions of P_μ and S_μ, the set of pairs P_μ, S_μ where both W^+ and W^- are space-like is open (and nonempty) and thus has positive measure in eight dimensions. I know of nothing precluding any pairs P_μ, S_μ arising from a Klein–Gordon wavefunction, so it seems reasonable to expect that the space–time points with space-like W^+ and W^- form a set of positive measure for many wavefunctions, perhaps for most.

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References

[1] Horton G, Dewdney C and Nesteruk A 2000 Time-like flows of energy momentum and particle trajectories for the Klein–Gordon equation *J. Phys. A: Math. Gen.* **33** 7337 (*Preprint* quant-ph/0103114)

[1] Two will not suffice for an example since P_μ and S_μ are linear combinations of the k_μ vectors.